

# No contradictions between Bohmian and quantum mechanics

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## Abstract

Two recent claims by A. Neumaier (quant-ph/0001011) and P. Ghose (quant-ph/0001024) that Bohmian mechanics is incompatible with quantum mechanics for correlations involving time are shown to be unfounded.

## 1 Introduction

Bohmian mechanics [1, 2] was proposed to give a realistic and deterministic interpretation of quantum mechanics. It assumes that every particle has, at any time  $t$ , a well-defined position on a trajectory that obeys Newton-type equations of motion. That motion is, however, influenced by a “quantum potential” which is related to the Schrödinger wave function of the particle. Trajectory parameters, although well-defined, cannot be known exactly. One can only know the probability density that the particle is at a given point, at a given time. That probability is equal to the absolute square of the normalized wave function. In this way Bohmian mechanics reproduces exactly the statistical predictions of quantum mechanics as regards expectation values of observables.

In recent papers, A. Neumaier [3] and P. Ghose [4, 5] have argued that although Bohmian mechanics may well exactly reproduce the single-time statistical predictions of quantum mechanics, the two approaches disagree on some multiple-time observables. It is the purpose of this note to show that these claims are unfounded.

## 2 Neumaier's argument

Neumaier [3] considers a one-dimensional harmonic oscillator of angular frequency  $\omega = 2\pi/T$ , in its ground state  $\psi_0$ . He then evaluates the following expectation value, in Bohmian and in quantum mechanics:

$$\langle \psi_0 | X(t_1 + T/2) X(t_1) | \psi_0 \rangle. \quad (1)$$

In quantum mechanics,  $X(t_1)$  is interpreted as the position operator in the Heisenberg picture. Since the position and momentum operators in the Heisenberg picture satisfy the same equations of motion as the classical position and momentum, one easily shows that  $X(t_1 + T/2) = -X(t_1)$ , so that

$$\langle \psi_0 | X(t_1 + T/2) X(t_1) | \psi_0 \rangle_Q = -\langle \psi_0 | X(t_1)^2 | \psi_0 \rangle_Q. \quad (2)$$

In Bohmian mechanics, on the other hand, the momentum associated with a real wave function (such as  $\psi_0$ ) always vanishes. The true value of position is constant, which means that  $x(t_1 + T/2) = x(t_1)$ . Hence

$$\begin{aligned} \langle \psi_0 | X(t_1 + T/2) X(t_1) | \psi_0 \rangle_B &= \int dx |\psi_0(x)|^2 x(t_1 + T/2) x(t_1) \\ &= \int dx |\psi_0(x)|^2 x(t_1)^2 \\ &= \langle \psi_0 | X(t_1)^2 | \psi_0 \rangle_Q. \end{aligned} \quad (3)$$

Since the right-hand side of (3) does not vanish, Eqs. (2) and (3) appear incompatible.

But are they? For this to be so, the expectation value (1) must have the same meaning in Bohmian as in quantum mechanics. This remark is anticipated in Neumaier [3]. However, spelling it out carefully shows, in our opinion, that his answer is inadequate.

In Bohmian mechanics, the meaning of (1) is straightforward. It is the average, over the ground state statistical ensemble, of the product of true values of the particle position at  $t_1$  and  $t_1 + T/2$ .

In Copenhagen quantum mechanics, the situation is more complicated since there is no such thing as the true value of position (except in the special case where the wave function is an eigenstate of the position operator). The meaning of (1) is related to the probability of measurement results. To see this, let the quantum state at  $t = 0$  be represented by a density operator

$\rho$ . Suppose that position measurements are made at  $t_1$  and  $t_1 + \tau$ . The probability of obtaining the results  $x$  at  $t_1$  and  $x'$  at  $t_1 + \tau$  is given by [6]:

$$P(x, t_1; x', t_1 + \tau) = \text{Tr} \left\{ |x'\rangle \langle x'| U(\tau) |x\rangle \langle x| U(t_1) \rho U^\dagger(t_1) |x\rangle \langle x| U^\dagger(\tau) \right\}. \quad (4)$$

Here  $U$  is the time-evolution operator. For simplicity, we have taken position to be discrete.

The average of the product of  $X$  (the position operator in the Schrödinger picture) at  $t_1 + \tau$  and  $X$  at  $t_1$  is given by

$$\begin{aligned} & \sum_x \sum_{x'} x x' P(x, t_1; x', t_1 + \tau) \\ &= \sum_x x \text{Tr} \left\{ \left[ \sum_{x'} x' |x'\rangle \langle x'| \right] U(\tau) |x\rangle \langle x| U(t_1) \rho U^\dagger(t_1) |x\rangle \langle x| U^\dagger(\tau) \right\} \\ &= \sum_x x \text{Tr} \left\{ U^\dagger(\tau) X U(\tau) |x\rangle \langle x| U(t_1) \rho U^\dagger(t_1) |x\rangle \langle x| \right\} \\ &= \sum_x x \langle x| U^\dagger(\tau) X U(\tau) |x\rangle \langle x| U(t_1) \rho U^\dagger(t_1) |x\rangle. \end{aligned} \quad (5)$$

Let  $\rho = |\psi_0\rangle \langle \psi_0|$  and let  $\tau = T/2$ , so that  $U^\dagger(\tau) X U(\tau) = -X$ . Then

$$\begin{aligned} \sum_x \sum_{x'} x x' P(x, t_1; x', t_1 + \tau) &= \sum_x x \langle x| -X |x\rangle \langle x| U(t_1) |\psi_0\rangle \langle \psi_0| U^\dagger(t_1) |x\rangle \\ &= \sum_x (-x^2) \langle \psi_0| U^\dagger(t_1) |x\rangle \langle x| U(t_1) |\psi_0\rangle \\ &= -\langle \psi_0| U^\dagger(t_1) X^2 U(t_1) |\psi_0\rangle \\ &= -\langle \psi_0| X(t_1)^2 |\psi_0\rangle. \end{aligned} \quad (6)$$

This establishes the meaning of (2) in terms of measurement results.

So we see that the expectation value (1) has a different meaning in Bohmian and in quantum mechanics. The average of true values cannot be calculated in Copenhagen quantum mechanics, and thus cannot be compared with the Bohmian result. On the other hand, probabilities of measurement results can be obtained in Bohmian mechanics. Successive measurements of  $X$  at  $t_1$  and  $t_1 + \tau$ , on a harmonic oscillator, can be represented as follows. Let  $|\alpha_0\rangle$  and  $|\beta_0\rangle$  be the initial states of two apparatus designed for position measurement. The initial state of the global system made of the harmonic oscillator and the two apparatus is given by

$$\Phi(t=0) = |\psi_0\rangle |\alpha_0\rangle |\beta_0\rangle = \sum_x \psi_0(x) |x\rangle |\alpha_0\rangle |\beta_0\rangle. \quad (7)$$

Between  $t = 0$  and  $t = t_1$ , the system evolves by an uninteresting phase factor. Indeed the three subsystems evolve independently, each being in an eigenstate of its own hamiltonian. At  $t = t_1$ , the first measurement is made (we take it to be instantaneous), so that the state is then ( $t_1^+$  means immediately after  $t_1$ )

$$\Phi(t = t_1^+) = \sum_x \psi_0(x) |x\rangle |\alpha_x\rangle |\beta_0\rangle, \quad (8)$$

where  $|\alpha_x\rangle$  is a pointer state corresponding to the value  $x$  of position. Between  $t_1$  and  $t_1 + \tau$ , the subsystems are decoupled, and the oscillator evolves following its own hamiltonian. Therefore,

$$\begin{aligned} \Phi[t = (t_1 + \tau)^-] &= \sum_x \psi_0(x) [U(\tau)|x\rangle] |\alpha_x\rangle |\beta_0\rangle \\ &= \sum_x \psi_0(x) \sum_{x'} [\langle x' | U(\tau) | x \rangle] |x'\rangle |\alpha_x\rangle |\beta_0\rangle. \end{aligned} \quad (9)$$

At  $t = t_1 + \tau$ , the second measurement is made and the state becomes

$$\Phi[t = (t_1 + \tau)^+] = \sum_x \psi_0(x) \sum_{x'} [\langle x' | U(\tau) | x \rangle] |x'\rangle |\alpha_x\rangle |\beta_{x'}\rangle. \quad (10)$$

The probability that the first measurement yields the value  $x$  and the second one yields  $x'$  is equal to the probability that the first pointer is truly at  $\alpha_x$  and the second one truly at  $\beta_{x'}$ . It is given by

$$|\psi_0(x) \langle x' | U(\tau) | x \rangle|^2. \quad (11)$$

This coincides with the right-hand side of Eq. (4). The alleged contradiction between (2) and (3) therefore disappears.

### 3 Ghose's argument

Ghose [4, 5] considers an experiment in which a pair of identical particles are simultaneously diffracted by two slits, as shown in Fig. 1. The particles may then be detected on a screen. We have  $L \gg a$ . We shall examine the form of the wave function near the screen, which is crucial to Ghose's argument.

Since the particles are taken to be bosons, the wave function at a specific time is given (up to a constant factor) as in Eq. (8) of Ref. [4]:

$$\Psi(x_1, y_1; x_2, y_2) = \psi_A(x_1, y_1) \psi_B(x_2, y_2) + \psi_A(x_2, y_2) \psi_B(x_1, y_1). \quad (12)$$

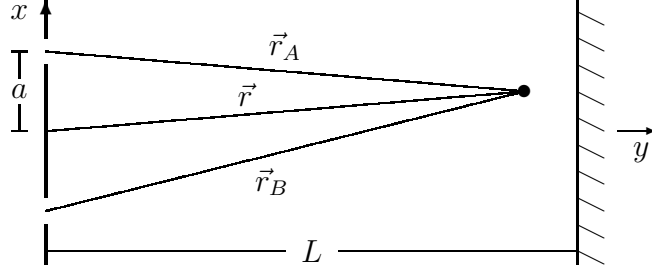


Figure 1: Two-slit interferometer.

The one-particle wave functions, however, should be represented by spherical, rather than plane waves. This corresponds to the fact that they originate from the slits. Referring to Fig. 1, we have

$$r_A = |\vec{r} - a\hat{x}| = \sqrt{y^2 + (x - a)^2} \approx y + \frac{1}{2L}(x - a)^2, \quad (13)$$

where the fact that  $y \approx L$  has been used, and higher order terms have been neglected. Similarly,

$$r_B \approx y + \frac{1}{2L}(x + a)^2. \quad (14)$$

For spherical waves we have

$$\psi_A(\vec{r}) = \frac{1}{r_A} \exp(ikr_A) \approx \frac{1}{L} \frac{1}{1 + \frac{y-L}{L}} \exp \left\{ ik \left[ y + \frac{1}{2L}(x - a)^2 \right] \right\}, \quad (15)$$

$$\psi_B(\vec{r}) = \frac{1}{r_B} \exp(ikr_B) \approx \frac{1}{L} \frac{1}{1 + \frac{y-L}{L}} \exp \left\{ ik \left[ y + \frac{1}{2L}(x + a)^2 \right] \right\}. \quad (16)$$

Substituting Eqs. (15) and (16) into (12), we get

$$\begin{aligned} \Psi = & \frac{1}{L^2} \left( 1 + \frac{y_1 - L}{L} \right)^{-1} \left( 1 + \frac{y_2 - L}{L} \right)^{-1} \exp \{ ik(y_1 + y_2) \} \\ & \cdot \left[ \exp \left\{ \frac{ik}{2L} [(x_1 - a)^2 + (x_2 + a)^2] \right\} \right. \\ & \left. + \exp \left\{ \frac{ik}{2L} [(x_1 + a)^2 + (x_2 - a)^2] \right\} \right]. \end{aligned} \quad (17)$$

From (17) we easily find that

$$\frac{dx_1}{dt} + \frac{dx_2}{dt} = \frac{\hbar}{m} \text{Im} \left\{ \frac{1}{\Psi} \left( \frac{\partial \Psi}{\partial x_1} + \frac{\partial \Psi}{\partial x_2} \right) \right\} = \frac{\hbar k}{mL} (x_1 + x_2). \quad (18)$$

Contrary to Ghose's claim,  $\dot{x}_1 + \dot{x}_2$  does not vanish. Therefore, his argument does not go through.

## References

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